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ORIGINAL RESEARCH

A Note on Fourier Transforms and Their Application

Tasmia Hoque[1]

[1] Department of Computer Science and Engineering, International Standard University, Dhaka, Bangladesh

Abstract

Fourier Transforms play an important role in applied sciences to better understand various physical phenomena. Especially in computational sciences, Fourier Transforms are very useful. In this project, we study continuous Fourier Transforms, Discrete Fourier Transforms, and their applications in science and engineering i.e. in noise reduction, image compression, Biomedical Engineering, COVID-19 data analysis and etc. Here, the applications of one dimensional and two-dimensional Fourier Transform have been described through noise filtering of signals and image processing. A few examples are illustrated to demonstrate the efficiency of Fourier Transforms. We implement the Fourier Transform in our real-life problems within the framework of MATLAB.

Many images that we use in our study have been taken from various internet sources to understand and apply DFT and FFT. We heartily acknowledge such anonymous help.

Keywords: Fourier Transform, Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT), Image Processing, Biomedical Engineering

1 | INTRODUCTION

Jean-Baptiste Joseph Fourier (1768–1830), who made substantial contributions to the study of trigonometric series, is commemorated with the name of the Fourier series. After initial research by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli, the Fourier Series is named after Jean-Baptiste Joseph Fourier. The formula for the Fourier Series coefficients was initially provided by Euler. Clairaut wrote what is now known as the first formula for the Discrete Fourier transform (DFT) in 1754, based on Euler's research. A DFT formula that does not rely on interpolating exclusively using odd or even periodic functions was published by Carl Friedrich Gauss (1777–1855) in 1805. The Fourier Transform and its name can be traced back to Joseph Fourier's (1768–1830) publication on heat flow in 1822.

Joseph Fourier wrote:

$$\phi(y) = a_0 \cos\frac{\pi y}{2} + a_1 cos3\frac{\pi y}{2} + a_2 \cos 5\frac{\pi y}{2} + \cdots$$

Multiplying both sides by $\cos(2k+1)\frac{\pi y}{2}$ and then integrating by $y=-1$ and $y=1$ Yields:

$$a_k = \int_{-1}^{1} \phi(y)\cos(2k+1)\frac{\pi y}{2}\, dy$$

This instantly provides any coefficient a_k of the trigonometrical series for $\phi(y)$ for any function that expands in this way. It is effective because, in the event if ϕ has such an expansion, the integral

$$= \int_{-1}^{1} a\cos\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2} + a_1\cos 3\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2}\, dy + \cdots$$

Double Fourier Series: The concept of a Fourier series development for a single-variable function x may be applied to the case of functions of two variables, x and y or $f\ (x,y)$. We can convert $f(x,y)$ into a double Fourier sine series, for instance

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{c_1}\sin\frac{n\pi y}{c_2}$$

Where,

$$B_{mn} = \frac{4}{c_1c_2}\int_0^{c_1}\int_0^{c_2} f(x,y)\ \sin\frac{m\pi x}{c_1}\sin\frac{n\pi y}{c_2}\, dxdy$$

Similarly, we can expand $f(x,y)$ into a double Fourier Cosine series

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos\frac{m\pi x}{c_1}\cos\frac{n\pi y}{c_2}$$

Now we move on to discuss some basics of Fourier Transforms. The rest of the paper is organized in the following way

- Fourier Transform and their preliminaries has been discussed.

- Application of image processing using two-dimensional Fourier Transform in different technologies of medical field has been explained.
- We finish this study with a short conclusion, future research directions & limitations.

1.1 | Fourier Transform

A Fourier Transform (FT) is a mathematical transformation that decomposes functions that depend on either time or space into functions that depend only on frequency. A technique known as the Fourier Transform divides a waveform (a function or signal) into a different representation that is defined by sine and cosine expressions with changing frequencies. Any waveform may be described as the sum of sinusoidal components, as the Fourier Transform demonstrates. We learned how to convert any periodic function into a sum of sinusoids using the Fourier Series. The application of this concept to a non-periodic function is the Fourier Transform. The Fourier Transform calculates the frequency of oscillations. Fourier Transform measures how much oscillations are at the frequency ω in the f. The mathematical form of the Fourier Transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

The Fourier Transform is applicable for non-periodic signals, but the Fourier series is only relevant to periodic signals. This is the distinction between the Fourier Transform and the Fourier Series.
Some of the properties of the Fourier Transform include:

1. It is a linear transform– If $g(t)$ and $h(t)$ are two Fourier Transforms given by $G(f)$ and $H(f)$ respectively, then the Fourier Transform of the linear combination of g and t can be easily calculated.
2. Time shift property– The magnitude of the spectrum shifts by the same amount in the Fourier Transform of $g(t-a)$, where $'a'$ is a real value and shifts the original function.
3. Modulation property– When a function is multiplied in time, another function modulates the original function.
4. Duality– The Fourier Transform of $G(t)$ is $g(-f)$ if $g(t)$ possesses the Fourier Transform $G(f)$.

1.2 | Fourier Integral

To figure out the Fourier transform is to use the Fourier integral. If a function $f(x)$ meets the Dirichlet conditions on each and every finite interval and is piecewise smooth on every interval $[-L,L]$, and if the integral $\int_{-\infty}^{\infty}|f(x)|dx$ converges then $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} \cos[u(x-t)\,du$ is known as the Fourier Integral. The second integral in the equation can be written as $\int_{-\infty}^{\infty} \cos[u(x-t)\,du = 2\int_{0}^{\infty} \cos[u(x-t)\,du$. So, another form of the Fourier Integral can be written as

$$\frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t)\cos[u(x-t)\,dt.$$

The Fourier Integral is very useful in the field of electrical communication, and it forms the basis of Cauchy's method for the solution of the partial differential equation.

1.2.1 | Advantages of Fourier Transform

A Fourier transformation's magnitude or width is expressed in terms of points. Both the frequency of values from the signal to be studied, as well as the number of values the conversion returned, are represented by the number of points employed in the transformation. The frequency resolution will be greater the more points that are employed in the transition. The Fourier Transform employs the entire waveform to transfer the signal into the frequency domain while maintaining information in amplitude, harmonics, and phase. The signal may be converted back into the time domain thanks to the Fourier transformation's preservation of phase information. An essential aspect of a signal that may be revealed by the Fourier Transform is its frequency components. By breaking up complex or noisy data into a sequence of trigonometric or exponential functions, it aims to make the data more understandable. Fourier transform is employed to ease transmissions and interpolate functions. Identifying frequency components is widely utilized in signal processing and represents one of the key tasks. It is utilized to break down the amplitude of a melodic composition into the relative intensities of each of its individual pitches. It is used in engineering to identify the main frequencies in a vibration analysis.

1.2.2 | Disadvantages of Fourier Transform

The fundamental tradeoff between frequency and temporal precision in the Fourier transform is its primary limitation. The Fourier transform cannot be applied to every unstable signal. Given that the signal's stability is the first requirement for the Fourier transform, in the realm of signals and systems, the Fourier transform has resulted in an extremely narrow and constrained perspective of frequency.

1.3 | Discrete Fourier Transform(DFT)

Decomposing a series of numbers into components with various frequencies yields the discrete Fourier Transform. The discrete Fourier Transform is used to finite evenly spaced sequences as follows: $Y_k = \sum_{n=0}^{N-1} X_n e^{-\frac{2\pi i}{N}} K_n$, where Y_k indicates the k^{th} element in the discrete Fourier Transform vector and X_n indicates the n^{th} element in the original time sequence. N is the number of elements in the time series. As defined by, the inverse discrete Fourier Transform is $X_n = \frac{1}{N}\sum_{n=1}^{N} Y_k e^{\frac{2\pi i}{N}} K_n$.
Think of an MP3 player connected to a speaker to get an idea of how the DFT accomplishes. Audio data from the MP3 player is sent to the speaker as variations in electrical signal voltage. The speaker drum vibrates as a result of those changes, and this in turn will cause air molecules to move and produce sound. The variations in an audio signal over time can be represented as a graph, where the x-axis represents time and the y-axis represents the electrical signal's voltage or even the mobility of the speaker's drum or air molecules. In either case, the signal appears as a squiggle-like irregular

wave. However, when we hear the music that was created by that squiggle, we can easily identify every instrument in a symphony orchestra, all of which were simultaneously playing distinct notes. This is due to the fact that the unpredictable squiggle is essentially the sum of several considerably more regular squiggles that stand in for various sound frequencies. Frequency simply refers to how quickly air molecules move back and forth or how quickly a voltage changes, and it may be shown as how quickly a regular squiggle moves up and down. When you combine two frequencies, the resultant squiggle moves upward where both of the component frequencies move downward where they both move downward, and it moves in the middle where they move in opposite directions. Decomposing a signal into its constituent frequencies is what the DFT performs mathematically, much like the human ear does physically. Some of the important applications of the Discrete Fourier Transform include

- Solving partial differential equations.
- Detection of targets from radar echoes.
- Correlation analysis.
- Computing polynomial multiplication.
- Spectral analysis.
- Convolutions and huge integer multiplications are examples of operations.
- Linear filtering etc.

1.4 | Fast Fourier Transform (FFT)

By matching together even and odd functions throughout calculation, it is possible to considerably improve the computation of the Discrete Fourier Transform as well as its Inverse Transform. The Fast Fourier Transform, which reduces the computing cost of the discrete Wavelet Transform from $O(N2)$ to $O(NlogN)$, is a combination of even and odd functions. It also has an inverse Fast Fourier Transform. The Discrete Fourier Transform of an arbitrary function is re-expressed using FFT. In the context of $N1$ smaller DFTs of sizes $N2$, consecutively, composite size $N = N1N2$ to make highly composite N's computation time $O(NlogN)$. The DFT matrix into a product of sparse (a matrix where most of the entries are zero) components, which then quickly computes transformations. The outcomes are obtained by the data being converted to the frequency domain using the FFT. The contemporary generic fast Fourier Transform algorithm was developed in 1965 and is commonly attributed to Cooley and John Tukey.

1. Fast multiplication of large integers and polynomials.
2. Effective multiplication of vectors and matrices.
3. Algorithms for filtering.
4. Discrete cosine or sine transforms with quick algorithms.
5. Chebyshev approximation quickly.
6. Computation of the distributions of isotopes.

2 | FOURIER TRANSFORM IN BIOMEDICAL ENGINEERING

A technique for resolving physical issues is the Fourier Transform. Numerous uses of the Fast Fourier Transform are found in the medical world. The Fast Fourier Transform (FFT) is utilized in medical image de-noising in a number of medical imaging modalities to recreate pictures from collected raw data. An effective method for simplifying the analysis of signals in the frequency domain is the Fast Fourier Transform. The core of digital signal processing is the Fast Fourier Transform (FFT). It is important in signal processing applications including de-noising, filtering, and compression.

2.1 | 2-D Image processing with FFT

We shall focus just on the Discrete Fourier Transform in this presentation as we are only interested in digital images (DFT). As the Discrete Fourier Transform (DFT) works by sampling the continuous Fourier Transform, it produces only a limited number of frequency values. Nonetheless, this number is typically sufficient to describe the image effectively in the spatial domain. The density of frequency components directly corresponds to the number of pixels in the original image, so the image size remains unchanged when moving between the spatial and frequency domains. In the case of a square image with dimensions $N \times N$, the two-dimensional DFT is expressed as: $F(k,l) = \frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b)e^{-2\pi i\left(\frac{ka}{N}+\frac{lb}{N}\right)}$

Here, the exponential term is the basis function equivalent towards each point $F(k,l)$ in the Fourier space, and $f(a,b)$ the picture in the spatial domain. According to the equation, each point's value (k,l) is determined by multiplying the spatial picture by the appropriate base function and adding the results.The fundamental operations are sine and cosine waves with progres sively higher frequencies, where $F(0,0)$ denotes the DC- component of the picture, which corresponds to average brightness, and $F(N-1, N-1)$ is the maximum frequency. The Fourier image can also be re-transformed into the spatial domain in a similar manner. The formula for the Inverse Fourier Transform is:

$$F(a,b) = \frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(k,l)e^{-2\pi i\left(\frac{ka}{N}+\frac{lb}{N}\right)}$$

To compute the values using the above equations, a double summation is required for each pixel in the image. However, since the Fourier Transform is separable, this process can be simplified. The transformation begins by applying N one-dimensional Fourier Transforms along one-dimension of the spatial image, resulting in an intermediate representation. Then, another set of N one-dimensional transforms is applied along the other dimension to produce the final frequency-domain image. This approach effectively reduces the computational load by converting the two-dimensional transform into a sequence of $2N$ one-dimensional transforms. Despite this optimization, computing the standard one-dimensional DFT remains mathematically intensive. If we use the Fast Fourier Transform (FFT) to estimate the one-dimensional DFTs, this may be condensed to $Nlog2N$. This is a major advancement, especially for huge photos. The size of the input picture that may be changed is often limited to $N = 2n$, in which n is an integer, by most FFT variants. The literature does a good job of describing the mathematical specifics. The output of the Fourier Transform is a complex-valued image, which can be represented in two ways: either by separating it into real and imaginary components, or by using its magnitude and phase. In image processing, it's common to display only the magnitude because it contains most of the structural information about the spatial domain image. However, if we plan to apply any operations in the frequency domain and then convert the image back to its original spatial form, both the magnitude and phase information must be preserved to ensure accurate reconstruction. Compared to a picture in the spatial domain, the Fourier domain image has a significantly wider range. Because of this, its values are often computed and stored as float values in order to be suitably precise.

2.2 | Fourier Transform in medical imaging

The Fourier Transform is a fundamental tool in image processing that breaks down an image into its sine and cosine components. While the original image exists in the spatial domain, the transformed result represents it in the frequency domain. frequency-based representation, each point corresponds to a specific frequency present in the spatial image. An image can be mathematically defined as some function $f(x, y)$, in which x and y are the spatial coordinates of the image and f (x, y) represents the brightness of the image at the point (x, y).Digital pictures, where $f(x, y)$ is a function having numerical values for x, y and brightness are used by computers. In digital photos, the brightness is related to as the gray level, and each component of the image is referred to as a picture element, or pixel, for short. A typical digital image could have a matrix with a set number of gray levels of 256 by 256 or more. The quantity of light that passes through an image-containing film is represented by the gray level. Several sectors of Fourier Transform in medical imaging:

- Plain X-Rays.
- MRI (Magnetic Resonance Imaging).
- CT (Computed Tomography).
- CAT (Computerized Axial Tomography)
- Chest Radiography.

2.3 | X-Ray

High-energy electromagnetic radiation with penetrating properties is known as an X-ray (X-radiation). The majority of X-rays have wavelengths between 10 picometers and 10 nanometers, which correspond to frequencies between 30 petahertz and 30 exahertz (310(16) Hz and 310(19) Hz). The most common uses of X-rays are to look for fractures (broken bones), identify pneumonia, and detect breast cancer. During X-Rays are sent from the source, the X-Ray gives a 2-D projection to the 3-D object. And then essentially, we get 2-D images. The X-Ray Beam loses some energy as it passes through the medium is $\ln(I) - \ln(I_0) = \int_{x_0}^{x} A(x)dx$, where I_0 is the initial intensity of beam and I is the final intensity of beam. For getting $A(x)$, the Radom Transform is needed. $R(f)(p, \theta) = \int_{-\infty}^{\infty} f\big(x(s), y(s)\big)ds$, where θ is the angle with x-axis and p is the distance from origin.

2.4 | The Radon Transform Theorem

An image matrix is projected along predetermined directions using the radon function. A collection of line integrals makes up a projection of the two-dimensional function $\boldsymbol{f(x,y)}$. By using parallel routes, or beams, in a certain direction, the radon method computes the line integrals from many sources. The dis tance between the beams is 1 pixel unit. The radon function rotates the source about the center of the picture, taking several parallel-beam projec tions of the image from various angles. The projection in the next image is shown at a certain rotation angle. For a function $\boldsymbol{f(p,\theta)}$ defined on $\boldsymbol{R^2}$ with compact support the Radon Transform of $\boldsymbol{f}$, denoted by $\boldsymbol{R(f)}$, is defined for $\boldsymbol{p} \in \boldsymbol{R}$ and $\boldsymbol{\theta} \in (\boldsymbol{0}, \boldsymbol{\pi}]$ as

$R(f)(p, \theta) = \int_{-\infty}^{\infty} f\big(x(t), y(t)\big)dt$, When the value zero is accepted outside of a compact set, a function has compact support. Given that we are only working with discrete regions (or slices) of an object, this is a realistic condition for a medical imaging task. Remember that we want to figure out the object's attenuation coefficient, which depends on its intensity. Since we are only working with finite slices, the attenuation coefficient must be equal to 0 outside of some limited area. The Fourier Slice Theorem explains that the one-dimensional Fourier Transform of a projection taken at an angle θ corresponds precisely to a line in the two-dimensional Fourier domain of the object, oriented at that same angle through the origin. This means that each projection provides a slice of the object's full frequency representation. By collecting projections at various angles, we can gradually assemble the complete frequency-domain image. Once the Fourier domain is fully constructed, applying a two-dimensional Inverse Fourier Transform enables us to recover the original object in the spatial domain. It serves as a connection between the two-dimensional and one-dimensional Fourier Transforms of the Radon Transform.

2.5 | Chest Radiography

A chest radiograph, also known as a chest X-ray (CXR) or chest film, is a radiographic projection of the chest used to identify problems affecting the chest, its contents, and adjacent structures. The most frequent film taken in medicine is a chest radiograph. Using Fourier Transform methods, a shift invariant model of the two-dimensional point spread response functions of the scattered radiation is deconvolved. No specialist imaging apparatus is needed for this method because it employs a digital radiograph obtained using a common chest imaging methodology. The shift variant form of the scatter model is optimal for the lung field, but when same model shape is applied to other chest areas, appropriate compensation is offered. Initial analysis indicates that this method can increase picture contrast over the whole chest area. The utilization of the FT and its inverse to eliminate unnecessary data from a picture:

1. An image created by combining the sine wave and chest radiography images, as well as the comparable Fourier spectrum in.

2. The undesired interference created by the sinusoidal brightness pattern may be eliminated by modifying the spatial frequency components, as indicated by the darker areas in the figure.
3. The original chest picture is then recovered using the inverse FT and is largely undistorted as demonstrated in.
4. Theoretically, further editing technique improvement would enable total visual quality reconstruction

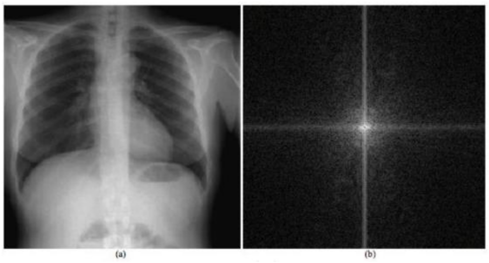

Figure 1: A Chest Radiograph and its 2-D Fourier spectrum are shown in (a) and (b)respectively.

The spatial frequency data display a wide range of values together with important vertical and horizontal characteristics connected to the vertebral column and ribs, respectively.

2.6 | MRI

Magnetic Resonance Imaging (MRI) is a diagnostic tool widely used by radiologists to visualize the body's internal structures and physiological processes. It uses powerful magnetic fields, gradient coils, and radiofrequency waves to generate detailed images of organs and tissues. MRI is commonly applied in hospitals and clinical settings for diagnosing diseases, monitoring treatment progress, and determining the stage of certain conditions. Compared to CT scans, MRI provides better contrast when imaging soft tissues, making it especially useful for examining areas like the brain and abdomen.

The Fourier transform, a basic mathematical method commonly employed in signal processing, is commonplace in radiology and essential to the creation of contemporary MR images. It is necessary to have a fundamental comprehension of the Fourier transform's functions in order to comprehend MRI procedures. The Fourier transform is the basis of several artifacts, including MR image encoding, filling of k-space, and other phenomena.

The Fourier Transform allows us to analyze the frequency components of complex signals. These signals can be visualized and even manipulated in the frequency domain. A well-known clinical application of this is MR spectroscopy, where data is displayed in the form of frequency and amplitude spectra. In this spectrum, each peak represents the presence and relative concentration of different biological metabolites within a specific region of interest (ROI). These metabolites resonate at slightly different frequencies, reflecting variations in their chemical structures.

2.7 | MR spectroscopy

MR spectroscopy, unlike conventional MRI, is designed to analyze the chemical composition of a focused region of interest (ROI) rather than producing anatomical images. It uses a specific radiofrequency pulse bandwidth to target this smaller area. Various neuronal metabolites such as myoinositol (ML), choline (Cho), creatine (Cr), glutamate and glutamine (Glx), N-acetyl aspartate (NAA), lactate (Lac), and lipids (Lip) resonate at distinct frequencies due to their unique chemical structures. The signal returned from the ROI is a complex combination of echoes from these metabolites. When processed through the Fourier Transform, this signal is separated into individual frequency components and their relative amplitudes. Since the Fourier Transform reflects only relative values, the term "relative" is essential, and each peak's height in the MR spectroscopy spectrum is meaningful only when compared to other peaks.

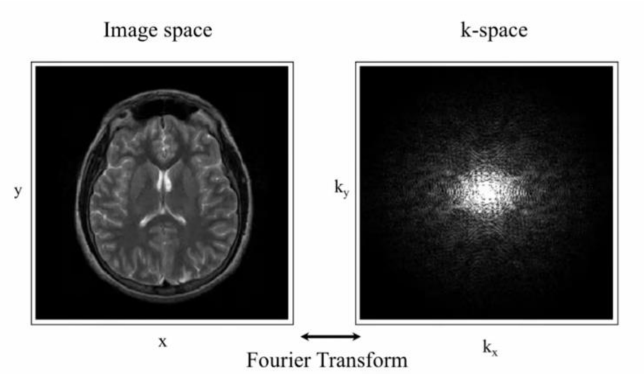

Figure 2: MR spectroscopy

In this coronal brain slice, spatial frequencies are sampled by progressively varying the magnetic field gradients (indicated by open arrows in the top three images) during the frequency and phase encoding steps. Although only three examples are illustrated, fully populating k-space requires many different combinations of gradient settings. Once k-space is completely filled, the final image is reconstructed by applying the inverse Fourier Transform, which essentially integrates the contributions of all spatial frequencies to form the complete image.

2.8 | K-SPACE

The spatial frequencies in the MR picture are represented by an array of integers called. $K -$ space. The mystery is maintained by the widespread visual portrayal of k-space as a "galaxy." In k-space, each "star" represents a single data point collected from the MR signal, with its brightness indicating the strength of that specific spatial frequency's contribution to the final image. Although k-space and the resulting MR image appear quite different, they actually contain the same underlying information about the scanned object. The two are mathematically linked through the Fourier Transform, which enables transformation between them.

Even though they look very different, the K-pace "galaxy" and MR picture both contain the same data about the scanned item. The two depictions might be utilizing a sophisticated mathematical technique (the Fourier Transform), transformed to one another. K-space is typically visualized as a rectangular grid defined by two main axes: k_x and k_y relating to the image's side-to-side and top-to-bottom spatial dimensions, respectively. Unlike the image axes that represent physical positions, these k-space axes represent spatial frequencies in the x and y dimensions. As such, there isn't a direct one-to-one correspondence between image pixels (x, y) and specific points (kx, ky) in k-space. Instead, each point in k-space contains information about the spatial frequency and phase that contributes to the entire image. Conversely, each pixel in the final MR image is influenced by the complete set of k-space data. This frequency-based representation of an MR image is conceptually similar to the diffraction patterns seen in X-ray crystallography, optical systems, or holography.

- $k_x$time signal
  y: diff. phase gradients
- FT in k_x: frequency in time (x-position)
  FT in k_y: frequency in phase (y-position)

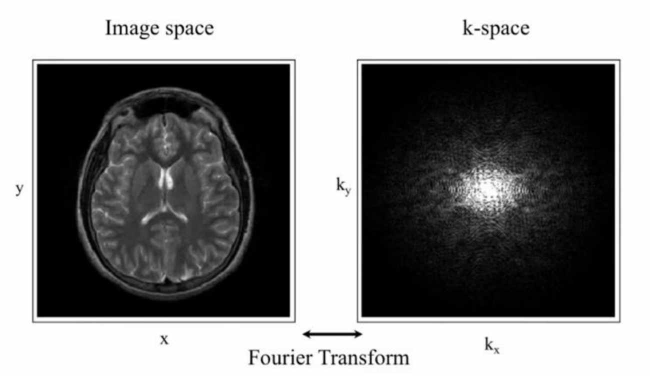

Figure 3: Every point in the picture and every point in k-space are mapped to one another.

Fourier Transform is a bridge connecting spatial space and spatial frequency space and the signal can be converted back and forth between these two spaces using the Fourier Transform.

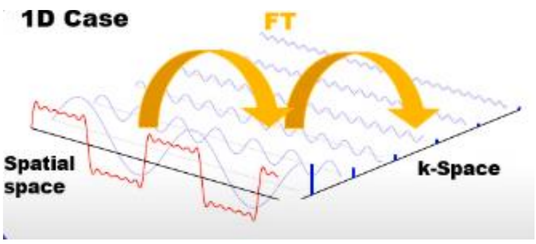

Figure 4: 1D illustration for Fourier Transform

The signal wave in red on the left is the square wave in spatial space by the Fourier Transform. The signal wave is decomposed into several trigonometric components with different spatial frequencies, and then these trigonometric components are assigned to k-space according to their spatial frequencies. That's how we get its k-space.

2.8.1 | Math of MRI

We need to know how MRI image formation works from a mathematical point of view. We place a person or any object to be scanned into a very large magnet, and this magnet aligns the spins of the hydrogen nuclei. Now this creates a net magnetization, and this is basically the goal of MRI. Firstly, we send in a radio wave that tips the magnetization into the transverse plane, and then in order to encode the position, we add a magnetic field gradient. The magnetic field gradient changes the relative phase of these spins. $M(x)e^{i\phi}$. Change in phase over change in time is frequency, so that means the phase is the integral of the frequency over time. B is just g times x nice linear gradient. If we assume that the person is not moving, so x is not varying with time. $G = \frac{dB}{dx}$ is the linear gradient in the magnetic field and in k-space $k = \gamma \int G\, dt$. Now we want to collect the signal with a radio frequency coil. Frequency coil sums up the spins over the entire space. Mathematically, we can represent that with an integral.

$$\begin{aligned} \phi &= \int \omega\, dt \\ &= \int \gamma B\, dt \\ &= \int \gamma G.x\, dt \\ &= x \int \gamma G\, dt \\ &= x.k \end{aligned}$$

Finally, we get, $M(x) = e^{ix.k}$.

And it is an equation of the Fourier Transform. So, by twisting up the spins using magnetic field gradients and summing them up using an RF coil, we have created a Fourier Transform, and if our goal is to get this m of x From the collected signal, we simply have to apply an Inverse Fourier Transform to the signal, and we can get the m of x.

2.9 | Expandation of 2d illustration for brain image

Firstly, we take the full transform of the spring image. It will first be decomposed into many components, and each component is the stripped image with that unique spatial frequency, then, they will be placed onto the k-space according to their spatial frequencies. After all components are placed correctly onto the k-space. The Fourier Transform is finished, and we get the k-space of this brain image.

Inversely, the inverse Fourier transform allows us to recover the brain image from its k-space. First, the points on the k-space will be converted back into many scraped images with different spatial frequencies according to their ease-based locations. Then by summing all these stripped images up, the image will be recovered.

- The majority of an image's low-frequency components are found in the middle regions of k-space, whereas its high-frequency components are found in the outside regions.
- The line linking a point at the center of the k-space is always perpendicular to the strapped patterns. The points on the x-axis correlate to vertical patterns, while the points on the y-axis correspond to horizontal patterns.

2.10 | Utilizing the number of COVID-19 daily deaths in Fourier analysis

In order to track the pandemic's behavior, the daily death toll is transformed to the frequency domain. The transformed peak spectrum may primarily reflect the intensity and progression of pandemic waves, rather than providing a direct measure of mortality.. Policymakers can use the frequency domain expression to determine whether to enhance their policies or not by understanding the pandemic cycle duration.

For real-time analysis, Fast Fourier Transform (FFT) technology is typically employed.

On Sunday, March 8, 2020, Bangladesh revealed the first three officially recognized instances of the coronavirus illness (COVID-19) in the nation. Few people are being affected by coronavirus so far. In Bangladesh, from 3 January 2020 to 17 October 2022, there have been 2,032,832 confirmed cases of COVID-19 with 29,402 deaths, reported to WHO. As of 10 October 2022, a total of 314,455,820 vaccine doses have been administered.

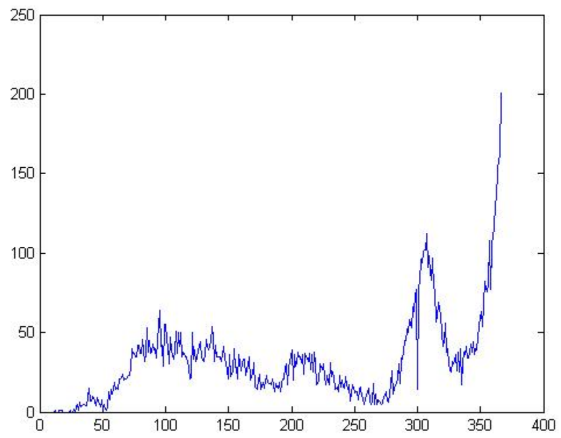

Figure 5: Time-Domain Graph

Figure 5 illustrates the daily death toll in Bangladesh, where the X-axis begins at '0' corresponding to March 8, 2020, and extends to '365', representing March 8, 2021.

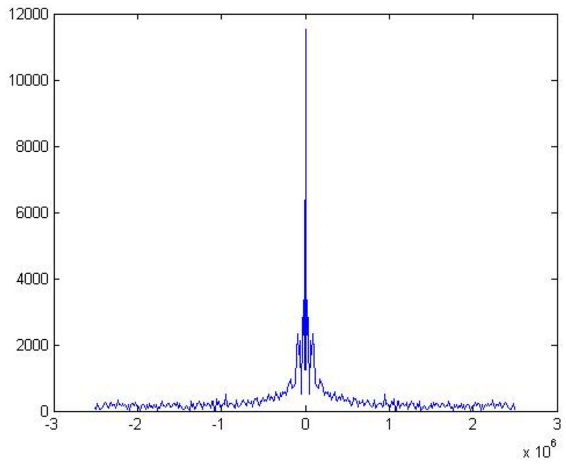

Figure 6: The outcome of an FFT-based spectrum analysis.

Figure 6 displays the findings of a spectrum analysis performed using 365 days between March 8, 2020, and March 8, 2021.

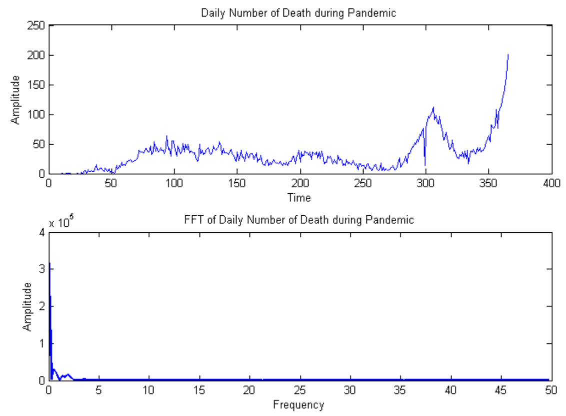

Figure 7: Fourier Transform of the daily number of deaths from March 8, 2020, to March 8, 2021.

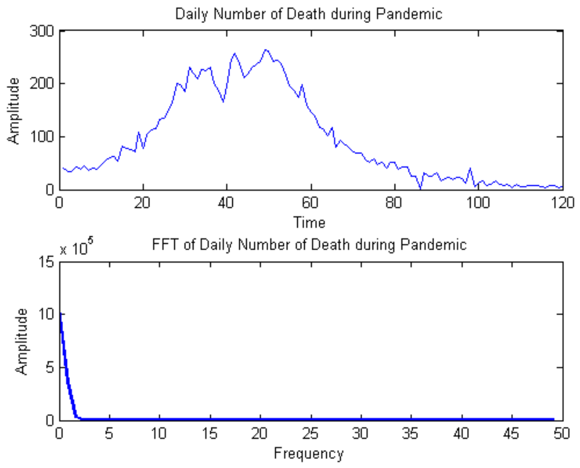

Figure 8: Fourier Transform of the daily number of deaths from June 2021 to October 2021.

The highest numbers of deaths were seen from June 2021 to October 2021 in Bangladesh. Figure 3.8 indicates higher amplitudes than Figure 3.6.

The computed spectrum is shown in Figure 3.9 using 365 days and different splitting days of 10 April, 30 June, 20 July, 29 August, 5 September, and 17 October.

1. Spectrum split on 10 April using the intervals [223:365] and [0:223]
2. Spectrum split on 30 June using the intervals [181:365] and [0:181]
3. Spectrum split on 20 July using the intervals [174:365]and [0:174]
4. Spectrum split 29 August using the intervals [134:365] and [0:134]
5. Spectrum split on 5 September using the intervals [114:365] and [0:114]
6. Spectrum split on 17 October using intervals [33:365] and [0:33]

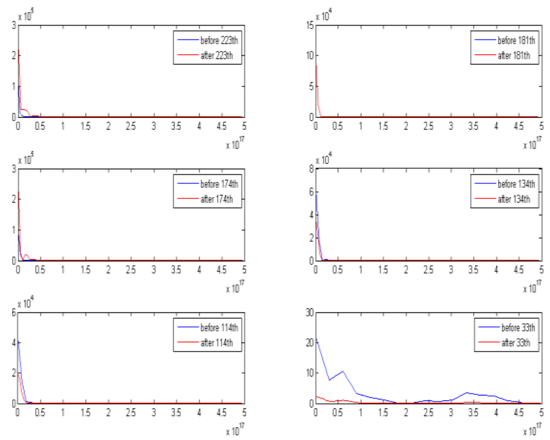

Figure 9: Comparison of power spectra before and after specific split days, showing variations in the frequency patterns of COVID-19 daily death data.

According to Figure 3.9a–f, the pandemic is suppressed during the longer period, whereas it recurs during the shorter cycle. The cycle duration utilizing the power peak spectrum provides a good description of the pandemic's propensity.

By applying FFT (Fast Fourier Transform) to the daily death data, even those new to the analysis can identify patterns in the pandemic's behavior or its cycle duration.
A longer cycle typically indicates that restrictions can be relaxed, whereas a shorter cycle signals the need for stricter measures to control the outbreak.

The ability of Fourier analysis to analyze epidemic wave breaks and provide meaningful statistics. We can discover the COVID-19 pandemic's hidden patterns by analyzing the frequency spectrum.

CONCLUSION

In this paper, noise removal of signals, noise reduction, image compression, MRI reconstruction, and COVID data analysis using FFT have been well studied. Using one one-dimensional FFT, a noisy signal has been manipulated and a denoised signal has been recovered. It can be used in noise cancellation software to remove annoying sounds and create harmonious sounds. FFT is also used in creating an Autoencoder to denoise ECG and radio signals. The effects of two-dimensional FFT in images have also been illustrated here. The image can be compressed to a certain limit without losing its identity with the help of the FFT. We notice that transforming noisy data from the time domain to the frequency domain and reducing Fourier coefficients with lower magnitude, large computation can be performed using limited storage. This idea has long been used for data compression and Sparse computations; Thus, the Fourier Transform has huge applications in various linear algebra packages and data science.

From this study, we are motivated to further research in Biomedical and seismic applications of the Fourier Transform.

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